25/11/08 Algebraic Geometry

Toric Varieties

(References: A. Kasprzyk, A Short Intro to TV)
Fulton's book
Cox, et al

§1 Introduction

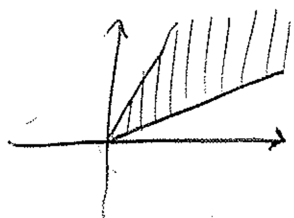
Let k be an algebraically closed field.
Then $\mathbb{T} = T = k^\times$ is the multiplicative group $k - \{0\}$
Then a torus is $\mathbb{T}^n = (k^\times)^n$
Let V be a variety over k.
Then V is toric if
(i) it contains a dense open subset $T \cong \mathbb{T}^n$ for some n
(ii) we have a group action
$$T \times V \longrightarrow V$$
which extends the obvious action $T \times T \longrightarrow T$

Example $\mathbb{P}^1 = \{[x_0, x_1] : x_0, x_1 \in k, \text{ not both zero}\}/[x_0,x_1] = \lambda[x_0,x_1],\ \lambda \in k^\times$
The obvious(?) choice for T is
$T = \{[1, x_1] \mid x_1 \neq 0\} \cong k^\times$ (open & dense)

Group action
$$\begin{array}{ccccc} T & \times & T & \longrightarrow & T \\ \cong & & \cong & & \cong \\ k^\times & \times & k^\times & \longrightarrow & k^\times \end{array}$$
So for $[1,a], [1,b] \in T$,
$$[1,a][1,b] = [1,ab]$$
We can extend this to the variety $V = \mathbb{P}^1$.
Let $[1,a] \in T$, $[x_0, x_1] \in V = \mathbb{P}^1$
Then $[1,a][x_0, x_1] = [x_0, a x_1]$ (well-defined)
so $\mathbb{P}^1$ is a toric variety!

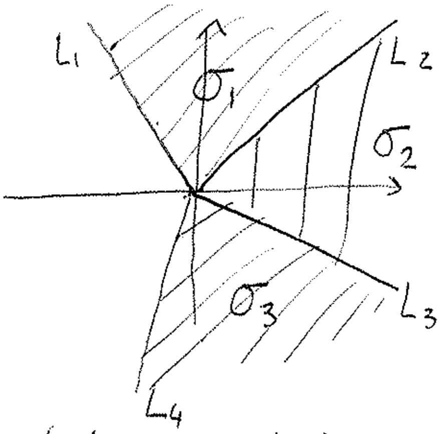
Other Examples of Toric Varieties
$\mathbb{A}^n$, $\mathbb{P}^n$, $V(x^3 - y^2) \subseteq \mathbb{A}^2$, etc., etc.

Fans A cone in k^n is something like:



(positive linear combinations of generators)

Definition A *fan* Δ is a collection of cones such that
(i) If $\sigma \in \Delta$, then $\sigma \cap (-\sigma) = \{0\}$
(ii) If $\sigma \in \Delta$ and τ is a face of σ then $\tau \in \Delta$
(iii) If $\sigma, \sigma' \in \Delta$, then $\sigma \cap \sigma' \in \Delta$



Then $\Delta = \{0, \sigma_1, \sigma_2, \sigma_3, L_1, L_2, L_3, L_4\}$

The nice fact about toric geometry is

$$\text{toric varieties} \xleftrightarrow{1-1} \text{fans}$$

§2 The fan associated to $\mathbb{P}^2$

$\mathbb{P}^2$ can be realised as three isomorphic copies of $\mathbb{A}^2$ glued together.
If $\mathbb{P}^2$ has coordinates $[x_0, x_1, x_2]$ we have the co-ordinate charts

$$U_i := \left\{ \left[\frac{z_0}{z_i}, \frac{z_1}{z_i}, \frac{z_2}{z_i} \right] : z_i \neq 0,\ i = 0,1,2 \right\}$$

e.g. $U_0 = \left\{ \left[1, \frac{z_1}{z_0}, \frac{z_2}{z_0}\right] \right\}$

These are glued together, e.g.

$$\phi_{0,1} : U_0 - \left\{ \frac{z_1}{z_0} = 0 \right\} \longrightarrow U_1 - \left\{ \frac{z_0}{z_1} = 0 \right\}$$

$$(1, x, y) \longmapsto \left(\frac{1}{x}, 1, \frac{1}{y} \right).$$

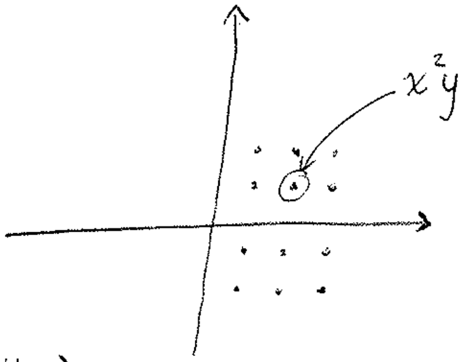
Write $x = \frac{z_1}{z_0}$, $y = \frac{z_2}{z_0}$, so, e.g. $U_0 = \{(1, x, y)\}$

Then the rings of regular functions are

$k[U_0] = k\left[\frac{z_1}{z_0}, \frac{z_2}{z_0}\right]$, $k[U_1] = k\left[\frac{z_0}{z_1}, \frac{z_2}{z_1}\right]$, $k[U_2] = k\left[\frac{z_0}{z_2}, \frac{z_1}{z_2}\right]$

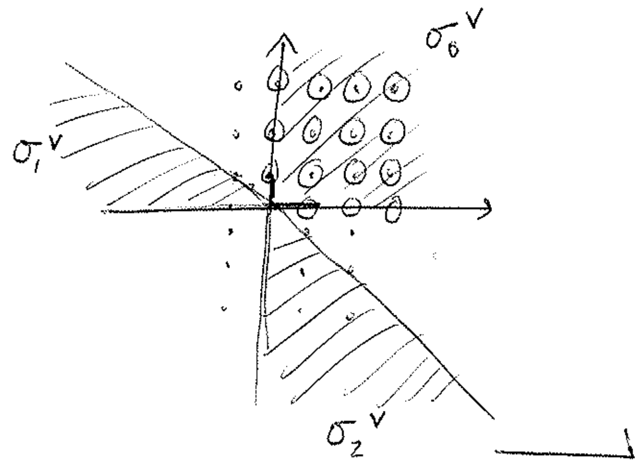
$k[U_0] = k[x, y]$, $k[U_1] = k[x^{-1}, x^{-1}y]$, $k[U_2] = k[y^{-1}, xy^{-1}]$

Let M = the lattice of $k[x^{\pm}, y^{\pm}] \cong \mathbb{Z}^2$
To each co-ordinate ring $k[U_i]$ we associate
a <u>cone</u> σ_i^{*}
which is a subset of $N = \mathrm{Hom}(M, \mathbb{Z}) \cong \mathbb{Z}^2$
Say $N = \langle h_1, h_2 \rangle$, $M = \langle e_1, e_2 \rangle$.
Then $h_i(e_j) = \delta_{ij}$
(to make rigorous, swap between k^n and $\mathbb{Z}^n$ by $\_\otimes k$ and $\_\cap \mathbb{Z}^n$)



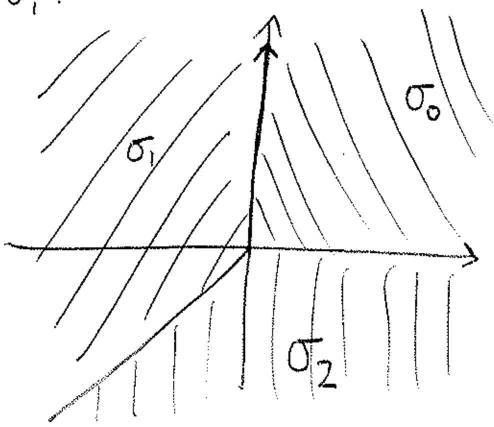
$$\sigma_i^{*} = \{ f \in N : f(u) \geq 0 \quad \forall u \in \sigma_i^{\vee} \}$$
(This makes sense, as $(\sigma^{\vee})^{\vee} = \sigma$)

[$\sigma_i^{\vee}$ is the cone in M given by the ring of regular functions:



Then σ_i have generators (as a semi-group)]
(finitely many, by Gordon's Lemma)
$\sigma_0 = \langle h_1 = (1,0), h_2 = (0,1) \rangle$
$\sigma_1 = \langle (0,1), (-1,-1) \rangle$
$\sigma_2 = \langle (1,0), (-1,-1) \rangle$

Draw σ_i:



Let $\Delta = \{ \sigma_0, \sigma_1, \sigma_2, \sigma_0 \cap \sigma_1, \sigma_0 \cap \sigma_2, \sigma_1 \cap \sigma_2, \sigma_0 \cap \sigma_1 \cap \sigma_2 \}$

Then Δ is the fan associated to $\mathbb{P}^2$.
Its structure reflects how $\mathbb{P}^2$ is glued together from affine pieces.